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Some Notes on the Measurement of Inequality, Poverty, Welfare and Discrimination

by

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Working Paper No.65: A Generalization of Gini based on Lorenz-Symmetry Considerations.

Working Paper No.66: An Index of Poverty and an Index of Inequality.

Working Paper No.67: On a Standard-of-Living Related Index of Aggregate Welfare.

Working Paper No.68: On Constructing an Index of Discrimination: A Brief Note.

A GENERALIZATION OF GINI BASED ON LORENZ-SYMMETRY CONSIDERATIONS

S.Subramanian*

ABSTRACT: This note develops, for the Gini coefficient of inequality, a very simple generalization that directly incorporates judjements on ' relative inter-group inequality aversion' by making the inequality measure sensitive to the skewness of the Lorenz curve.

*ACKNOWLEDGEMENT: The inequality measure $\tilde{G}$ mentoned as a 'special case' in section 2 of this note was suggested to the author by Dr.M.N.Murthy in conversation: his suggestion has played a central role in motivating this entire note. Thanks are also due to Dr. Murthy for useful suggestions on an earlier draft of this paper. Errors are solely the author's responsibility.

A GENERALISATION OF GINI BASED ON LORENZ- SYMMETRY CONSIDERATIONS

1. INTRODUCTION

A well-known difficulty with the Gini ratio (G) of inequality is that so long as the area enclosed by the diagonal of the unit square and the Lorenz curve is the same for any given set of Lorenz curves, G will rank the distributions represented by the different Lorenz curves indifferently -- without regard for the skewness of the Lorenz curves. If inequality among the poorer income groups is regarded as more pernicious than inequality among the richer income groups -- and this seems to be a reasonable position to adopt -- then it makes intuitive sense to penalize a distribution for which the Lorenz curve is skewed toward (1,1) of the unit square relative to a distribution for which the Lorenz curve is skewed toward (0,0). In what follows, we develop a measure of inequality $G(\lambda)$, where λ is in the nature of an 'index of inter-group inequality aversion'; for parametric variations in λ (within bounds to be discussed in the next section), $G(\cdot)$ exhibits varying degrees of sensitivity to the skewness of the Lorenz curve.[1] The Gini ratio G becomes just a special case of $G(\lambda)$.

2. A GENERALIZATION OF GINI

X is a random variate designating (say) income, and is distributed with mean μ. $F(\cdot)$ is the cumulative density

function, and H(.) the first-moment distribution function of X, with

$F(x) = \int_0^x F'(X)\,dX$ and

$H(x) = \frac{1}{\mu} \int_0^x XF'(X)\,dX.$

It is to be noted that $\lim_{x \to 0} F(x) = \lim_{x \to 0} H(x) = 0$, and $\lim_{x \to \infty} F(X) = \lim_{x \to \infty} H(x) = 1.$

The Lorenz curve is simply the functional relationship between F(x) and H(x), and is drawn in Figure 1.

Now let x* be the income level corresponding to which the Lorenz curve (see Figure 1) intersects the diagonal drawn from (0,1) to (1,0) of the unit square. (This diagonal is referred to by Kakwani (1980) as the 'alternative diagonal'). The cumulative proportion of population with incomes no higher than x* is denoted by F*, and the cumulative share in total income of those with incomes no higher than x* is denoted by H*.

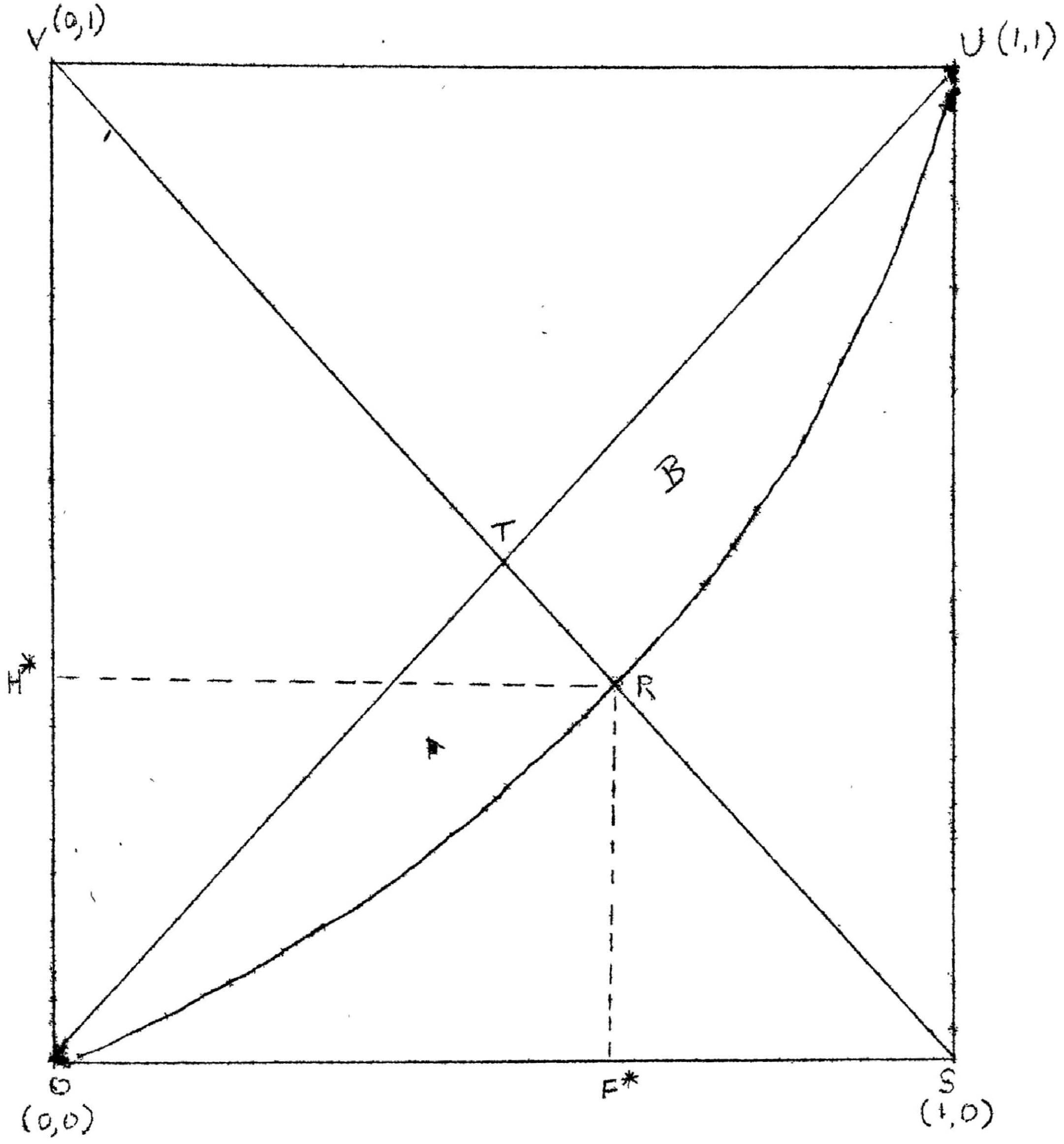
The Gini ratio of inequality in the distribution of income is given by the following well-known expression:

$$G = 1 - 2\int_0^{\infty} H(x)F'(x)\,dx$$

$$= 1 - 2\left(\int_0^{x^*} H(x)F'(x)\,dx + \int_{x^*}^{\infty} H(x)F'(x)\,dx \right). \quad \ldots(1)$$

Letting G_1 and G_2 stand for the Gimi ratios of inequality in the distribution of income among, respectively, those with incomes less than or equal to x* and those with incomes

Figure 1: The Lorenz Curve



higher than x*, it is routine to note that

$G_1 = 1 - 2\int_0^{x^*} H(x)F'(x)dx / F^*H^*$, so that

$$2\int_0^{x^*} H(x)F'(x)dx = (1-G_1)F^*H^* \qquad ...(2a)$$

and

$G_2 = 1 - 2\{\int_{x^*}^{\infty} H(x)F'(x)dx - H^*(1-F^*)\}/(1-F^*)(1-H^*)$, so that

$$2\int_{x^*}^{\infty} H(x)F'(x)dx = (1-G_2)(1-F^*)(1-H^*) + 2H^*(1-F^*). \qquad ...(2b)$$

From (2a) and (2b), and making use of the fact that since (F^*,H^*) is a point on the alternative diagonal of the unit square, so that $F^*+H^*\equiv 1$, one has:

$$2\int_0^{x^*} H(x)F'(x)dx = F^*(1-F^*)(1-G_1); \text{ and}$$
$$2\int_{x^*}^{\infty} H(x)F'(x)dx = F^*(1-F^*)(1-G_2) + 2(1-F^*)^2. \qquad ...(3)$$

Making the appropriate substitutions from (3) into (1) yields

$$G = 1 - \{F^*(1-F^*)(1-G_1) + F^*(1-F^*)(1-G_2) + 2(1-F^*)^2\}$$

which, upon simplification, can be written as

$$G = 1-2(1-F^*)\{1-F^*(\tfrac{1}{2}G_1+\tfrac{1}{2}G_2)\}. \qquad ...(4)$$

Writing G in the form of (4) above has the advantage of bringing sharply into focus the fact that Gini attaches an identical proportionate weightage (of one-half) to both G_1 and G_2. This paves the way for a very natural generalization of G to a measure that we may call $G(\lambda)$, and which is given by

$$G(\lambda) = 1 - 2(1-F^*)\{1-F^*(\lambda G_1 + (1-\lambda)G_2)\},\ \lambda \in \left[\tfrac{1}{2}, 1\right]$$

λ may be regarded as an index of 'relative inter-group inequality aversion', and the closer λ is to unity, the greater the relative concern for inequality among the 'poorer' income groups. At the extreme of $\lambda=1$, one is expressing the judgement that inequality among those with incomes exceeding x^* is of no account at all in assessing the extent of overall inequality among the entire population.

It is immediately clear that

for $\lambda=1/2$, $G(\lambda)=G$; and

for $\lambda>1/2$, $G(\lambda)=2F^*(1-F^*)(\lambda-\frac{1}{2})(G_1-G_2) \gtreqless 0$ as $G_1 \gtreqless G_2$. ...(6)

From (6) we note that when $\lambda=1/2$, G becomes just a special case of $G(\lambda)$; for $\lambda>1/2$, $G(\lambda)$ exceeds G if the Lorenz curve is skewed toward (1,1) of the unit square, is less than G if the Lorenz curve is skewed toward (0,0), and coincides with G if the Lorenz curve is symmetric.

One way of 'correcting' for the skewness of the Lorenz curve -- i.e. of ensuring that the index of inequality is greater than, equal to, or less than G according as the Lorenz curve is skewed toward (1,1), symmetric, or skewed toward (0,0) -- is by employing a particularly simple and intuitively appealing measure of inequality given by

$\tilde{G}=G+(A-B)$, ...(7)

where A is the area ORT in Figure 1 and B the area TRU in Figure 1. We take note of -- without proof -- the interesting fact that $\tilde{G}$, like G, is also a special case of $G(\lambda)$; in particular,

for $\lambda = 3/4$, $G(\lambda)=\tilde{G}$.

3. AN EMPIRICAL ILLUSTRATION

Kakwani (1980) has suggested an alternative coordinate system for the Lorenz curve: the Lorenz curve, which plots the relationship between H(x) and F(x), can also be defined by the relation

$E(x) = g(P(x))$

where E^2 and P^2 are defined as:

$E = \frac{1}{\sqrt{2}}(F-H)$; and

$P = \frac{1}{\sqrt{2}}(F+H)$.

It is to be noted that E(x) is the perpendicular distance from the diagonal of the unit square to the point on the Lorenz curve corresponding to an income level of x, and P(x) the distance from the origin, along the diagonal, to the point from which the perpendicular is drawn. Hence P will vary from 0 to $\sqrt{2}$, and at the two extreme values of P, E will be zero. A specific functional form of g which has been noted to yield good estimation results is given by

$g(P) = aP^{\alpha}(\sqrt{2}-P)^{\beta}$; $a>0$, $0<\alpha\leq 1$, $0<\beta\leq 1$.

Of interest in the present context is the fact that the ratio α/β is a measure of the skewness of the Lorenz curve: $\alpha/\beta \gtreqless 1$ according as the Lorenz curve is skewed toward (1,1), symmetric, or skewed toward (0,0).

For fifty countries, Kakwani(1980) has presented OLS parameter estimates of the alternate coordinate system for the Lorenz curve; for these countries, he has also computed the Gini index and the measure of skewness α/β. For illustrative purposes we present in Table 1 below his results for two countries: Taiwan and Australia.

Table 1: Parameter Estimates of the Alternative Coordinate System for the Lorenz Curve, Gini Index and a Measure of Skewness in Two Countries.

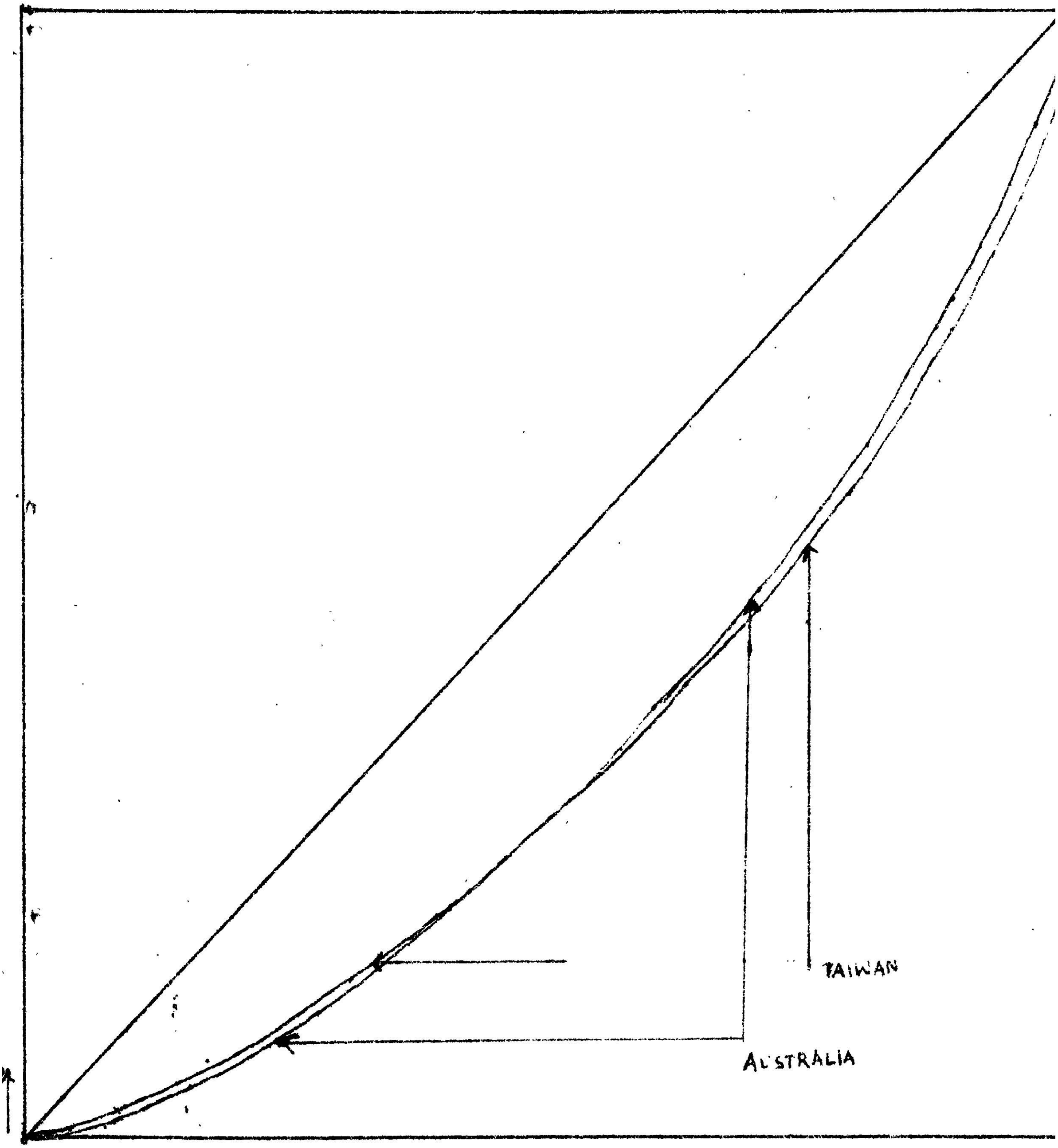
Country	Year	Population	a	α	β	G	α/β
Taiwan	1964	Households	.282	.817	.754	.329	1.084
Australia	1967/68	Households	.274	.763	.818	.318	.933

Source: From Tables I7.1 and I7.2 of Kakwani (1980), pp.386-89.

It is clear from the α/β ratios in the Table above that the Lorenz curve for Taiwan is skewed toward (0,0) and for Australia toward (1,1). Visual confirmation of this is available in Figure 2[3].

Now while Taiwan is ranked by G above Australia (in descending order of inequality), it is interesting to ask if there is any value of λ for which $G(\lambda)$ reverses the ranking by G. Table 2 below presents some additional cal-

FIGURE 2: LORENZ CURVES OF INCOME DISTRIBUTION AMONG HOUSEHOLDS IN TAIWAN (1964) AND AUSTRALIA (1967/68).



culations we have performed using Kakwani's parameter estimates. (F^*, in the Table below, is derived from the relation $F = (P+E)/\sqrt{2}$; it is immediate that for the point of intersection of the Lorenz curve with the alternative diagonal, $P = 1/\sqrt{2}$ and $E = a\left(\frac{1}{\sqrt{2}}\right)^{\alpha}\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)^{\beta} = a\left(\frac{1}{\sqrt{2}}\right)^{\alpha+\beta}$, so that $F^* = \dfrac{\frac{1}{\sqrt{2}}+a\left(\frac{1}{\sqrt{2}}\right)^{\alpha+\beta}}{\sqrt{2}}$).

Table 2: F^*, G_1, and G_2 for two countries

Country	Year	Population	F^*	G_1	G_2
Taiwan	1964	Households	.616	.183	.230
Australia	1967/68	Households	.612	.212	.184

Letting the subscripts T and A stand for Taiwan and Australia respectively, and making use of (5), it is a matter of routine manipulation to note that the value of λ for which $G_T(\lambda)$ will equal $G_A(\lambda)$ is given by

$$\lambda^* = \frac{F_T^*\{1+(1-F_T^*)G_{2T}\} - F_A^*\{1+(1-F_A^*)G_{2A}\}}{F_A^*(1-F_A^*)(G_{1A}-G_{2A}) - F_T^*(1-F_T^*)(G_{1T}-G_{2T})}.$$

Making the appropriate substitutions from the data presented in Table 2 into the right hand side of the equation above, we obtain $\lambda^*=.83$; and for $\lambda=.83$, $G_T(\lambda)=G_A(\lambda)=.322$. Briefly, for any value of $\lambda \varepsilon [1/2, .83)$, $G(\lambda)$ will pronounce the Taiwanese distributuon to be more unequal than the Australian distribution, while for any value of $\lambda \varepsilon (.83, 1]$, $G(\lambda)$

will pronounce the Australian distribution to be more unequal than the Taiwanese. It is, ultimately, a question of what relative weightage one wishes to attach to inequality among the poorer income groups *vis-a-vis* the richer groups.

4. CONCLUDING OBSERVATIONS

In this note, a simple and natural generalization $G(\lambda)$ of the Gini ratio G of inequality has been discussed. While Gini is insensitive to whether inequality is more pronounced in the lower or the upper end of the distribution, its generalized form $G(\lambda)$ permits judgements on 'relative inter-group inequality aversion' to be directly incorporated in the inequality measure. Apart from G, a second special case $\tilde{G}$ of $G(\lambda)$ has also been presented in this note. Finally, an empirical illustration has been provided to demonstrate how the inequality ranking by G could -- for a sufficiently acute degree of sensitivity to inequality among the poorer income groups -- be reversed by $G(\lambda)$.

NOTES

1. For a parametrization of Gini along different lines, see Yitzhaki (1983).
2. Since there is no ambiguity, we omit to specify yhe argument x of E and P.
3. Figure 2 represents a free-hand curve fit based on the OLS parameter estimates presented in Table 1.

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Yitzhaki, Shlomo. (1983): 'ON an Extension of the Gini Inequality Index', International Economic Review, Vol. 24, No. 3, October 1983, 617-628.

ON AN INDEX OF POVERTY AND AN INDEX OF INEQUALITY

S.Subramanian

Abstract : This note introduces a measure of poverty which is a variant of the Sen index, and a measure of inequality which is a variant of the Gini index. The new indices possess some attractive 'transfer-sensitivity' properties.

1. INTRODUCTION.

In this note, I propose a simple variant of the Sen(1976) index of poverty which, for a particular pair of values of the poverty line and the head-count ratio, collapse to a measure of inequality that is relatively more sensitive to inequality among the lower income groups than is the Gini index of inequality. The axiomatic system that is used to derive the index of poverty in this note is -- while being inspired by the Sen system -- both more economical than and different from the one employed by Sen. In particular, (i) the specification of the functional form of the poverty measure is different; (ii) the 'normalization' axiom employed by Sen is replaced by one due to Nayak(1985); and (iii) Sen's 'rank order weighting' axiom is dispensed with. The resulting poverty index also betrays a fair measure of success in meeting the requirements of Kakwani's (1980) transfer sensitivity axioms, unlike the Sen index.

2. NOTATION.

$\underline{x} = (x_1, \ldots, x_i, \ldots, x_n)$ is an n-vector of individual incomes

arranged in non-decreasing order: $x_i \leq x_{i+1}$ $\forall i \in S$, where S is the set of all individuals. The poverty line, which is a level of income below which people are judged to be absolutely impoverished, is denoted by z (>0). The set of all people with incomes not exceeding the poverty line is S^P, and the cardinality of this set is q. $H \equiv q/n$ is the proportion of people in poverty, or the head-count ratio. μ^P and μ are, respectively, the average incomes of the poor and of the entire population, viz. $\mu^P \equiv \frac{1}{q} \sum_{i \in S^P} x_i$, and

$\mu \equiv \frac{1}{n} \sum_{i \in S} x_i$. $I \equiv 1 - \mu^P/z$ is the proportionate poverty gap, that is, the shortfall of the average income of the poor from the poverty line, expressed as a ratio of the latter. $\mu_i \equiv \frac{1}{i} \sum_{j=1}^{i} x_j$ is the average of the incomes of those with incomes less than or equal to the i th poorest person's income x_i. For terminological convenience -- but without necessarily regard for terminological exactitude -- I shall refer to [illegible] μ_i as 'person i's mean income'. Finally, D^P is a measure of inequality in the distribution of income among the poor, and is given (for large q) by $D^P = 1 - \sum_{i \in S^P} \mu_i / q\mu^P$.

(This inequality measure will be discussed later).

3. RECALL: SEN'S INDEX OF POVERTY.

Sen (1976) employed the following system of axioms to derive his poverty measure (P_S).

Axiom 1 (Specification of Functional Form). The poverty measure is a weighted sum of the shortfalls of the incomes of the poor from the poverty line:

$$P_S = A(z,q,n)\sum_{i \in S^P} v_i(z,\underline{x})(z-x_i),$$

where $A(z,q,n)$ is a 'normalizing' parameter, and $v_i(z,\underline{x})$ is the weight attached to the ith poorest person's income shortfall.

Axiom 2 (Rank- Order Weighting). The weight $v_i(z,\underline{x})$ on the ith poorest person's income shortfall is given by the person's rank in the income vector of the poor:

$$v_i(z,\underline{x}) = q+1-i.$$

Axiom 3 (Normalization'). When income among the poor is equally distributed, the poverty measure is given by P_S=HI.

The following theorem is now true.

Theorem 1. For large numbers of the poor, the only poverty measure satisfying Axioms 1, 2 and 3 is givem by

$$P_S = H\{I+(1-I)G^P\}$$

where G^P is the Gini ratio of inequality in the distribution of incomes among the poor.

Proof. Sen(1976). □

4. AN ALTERNATIVE AXIOMATIC SYSTEM.

In what follows, I present an alternative set of axioms to derive a variant P of the Sen index of poverty P_S.

Axiom ∝ (Specification of Functional Form). The poverty

measure P is the sum, over all individuals among the poor, of the shortfall from the poverty line of each person's mean income:

$$P = A(z,n)\sum_{i \in S^P} (z-\mu_i) \qquad \ldots(1)$$

where A(z,n) is a normalizing parameter.

Axiom β (Normalization). As the mean income of the entire population tends to zero (so that the head-count ratio tends to unity), the poverty measure tends to unity.

Remark 1: While Axiom 1 presents the poverty measure as a wighted sum of the individual shortfalls of incomes from the poverty line, in Axiom ∝ the form of the poverty measure is specified differently: the community of n individuals is divided into n distinct groups of individuals, with group i being constituted by the i poorest individuals; the poverty measure is then presented as the (unwighted) sum of the shortfalls in the average incomes of all groups from the poverty line.

Remark 2: The normalization axiom above -- Axiom β -- is borrowed from Nayak(1985), who makes the interesting point that Sen's Axiom 3 is, properly speaking, not a normalization axiom at all. Since the normalization of a function is 'a method of fixing the scale of its range values'[1], for a real-valued function such as P_s(or P), all that is required for normalization of the function is the fixing of its range-values for any two points in the function's

domain. From Axiom 1 (and also Axiom ∝) it is clrar that when the head-count ratio is zero, the poverty measure is also zero. For purposes of normalizing the poverty function, therefore, it is sufficient to fix its range-value for just one more value of the head-count ratio -- and this is precisely what Axiom β does. This must be contrasted with Axiom 3 in which the range-value of the poverty function is fixed for every value of the head-count ratio in the closed interval [0,1].

The following Theorem is now true.

Theorem 2. For large numbers of the poor, the only poverty measure satisfying Axioms ∝ and β is given by

$$P = H\{I+(1-I)D^P\}$$

where D^P is the index of inequality (introduced in section 1) in the distribution of incomes among the poor.[2]

Proof. Axioms ∝ and β together imply that

$$1 = A(z,n)nz,$$

so that

$$A(z,n) = 1/nz.$$

Substituting for A(z,n) from above into (1) yields

$$P = \frac{1}{nz}\sum_{i\varepsilon S^P}(z-\mu_i) = \frac{q}{n} - \frac{1}{nz}\sum_{i\varepsilon S^P}\mu_i \ .$$

From the definition of D^P we know that $\sum_{i\varepsilon S^P}\mu_i = q\mu^P(1-D^P)$,

so that, given (2), one has

$$P = \frac{q}{n} - \frac{1}{nz}\{q\mu^P(1-D^P)\}$$

$$= \frac{q}{n}\{1- \frac{\mu^P}{z}(1-D^P)\}. \qquad \ldots(3)$$

Recalling that $q/n \equiv H$ and $1-\mu^P/z \equiv I$, (3) can be written as

$$P = H\{I+(1-I)D^P\}. \;\Box$$

5. POVERTY MEASUREMENT AND TRANSFER-SENSITIVITY.

The poverty measure derived in section 4 above is identical to the Sen index of poverty save that the Gini ratio of inequality among the poor, G^P, has in the measure P been replaced by the measure of inequality D^P. The effect of this is that the poverty index P -- unlike the Sen index P_S -- satisfies (in the one case unconditionally and in the other conditionally) the following two 'transfer-sensitivity' axioms proposed by Kakwani (1980).

Transfer-Sensitivity Axiom I (or Axiom TS-I). If $(\Delta K)i,i+\sigma$ is the increase in the poverty measure K due to a transfer of income h from the ith poorest person to the $(i+\sigma)$th poorest person, with σ being any positive integer, then $(\Delta K)i,i+\sigma > (\Delta K)j,j+\sigma$ for $j > i$.

Transfer-Sensitivity Axiom II (or Axiom TS-II). If $(\Delta K)i,j$ is the increase in the poverty measure K due to a transfer of income h from the i th poorest person, who has income x_i, to the jth poorest person who has income $x_i+\delta$, where δ is any positive constant, then $(\Delta K)i,j > (\Delta K)m,n$ for $m>i$ and $n>j$.

That the poverty index satisfies Axiom TS-I can be readily seen by noting that

$$(\Delta P)_{i,i+\sigma} = \frac{H(1-I)\sigma h}{\mu^P i(i+\sigma)} \text{ and } (\Delta P)_{j,j+\sigma} = \frac{H(1-I)\sigma h}{\mu^P j(j+\sigma)}, \text{ so that}$$

$$(\Delta P)_{i,i+\sigma} - (\Delta P)_{j,j+\sigma} = \frac{\sigma Hh(1-I)(j-i)(j+i+\sigma)}{\mu^P ji(i+\sigma)(j+\sigma)} > 0 \text{ since } j>i.$$

P, however, may or may not satisfy Axiom TS-II. To see the condition under which P will satisfy Axiom TS-II, note that

$$(\Delta P)_{i,j} = \frac{H(1-I)h}{\mu^P}\left(\frac{j-i}{ij}\right) \text{ and } (\Delta P)_{m,n} = \frac{H(1-I)h}{\mu^P}\left(\frac{n-m}{mn}\right), \text{ so that}$$

$$(\Delta P)_{i,j} - (\Delta P)_{m,n} = \frac{H(1-I)h}{\mu^P}\left\{\frac{j-i}{ij} - \frac{n-m}{mn}\right\} > 0 \text{ if, and only if,}$$

$$(j-i) > \left(\frac{ij}{mn}\right)(n-m).$$

Notice that for $m \gg i$ and $n \gg j$, the ratio (ij/mn) becomes vanishingly small, so that, under these circumstances, Axiom TS-II will be rather easily satisfied. However, the Axiom is not invariably satisfied -- and it is also not readily clear that it is desirable for the Axiom to <u>be</u> invariably satisfied. The notion of relative deprivation, it can be argued, is <u>also</u> mediated by the number of people that are wedged in between the two individuals involved in a transfer of income; and Axiom TS-II does not seem to take

sufficient note of this. A simple example should clarify this point.

Consider two alternative situations in the first of which i=2, j=3, m=4 and n=100, and in the second, i=2, j=3, m=99 and n=100. (In both cases, $x_j - x_i = x_n - x_m \doteq \delta$). In case (i), $(j-i)\ \{=1\} \ngtr \left(\frac{ij}{mn}\right)(n-m)\ \{\doteq 1.44\}$, while in case (ii), $(j-i)\{=1\} > \left(\frac{ij}{mn}\right)(n-m)\ \{\doteq .0006\}$.

In case (i), TS-II is violated, and in case (ii), TS-II is satisfied. This does not accord ill with one's intuition which, in assessing the desirable impact of income transfers on poverty reduction, does permit a trade-off between the position in the rank-order of, and the inter-personal positional distance between, those involved in the income transfers.

It should be mentioned here that while the Sen index of poverty violates both Axioms TS-I and TS-II there exists a class of poverty measures suggested by Kakwani (1980) that (conditionally) satisfy the two Axioms. This class of poverty measures is obtained by replacing Sen's Axiom 2 (Rank-Order Weighting) with the following Axiom:

Axiom 2'(Generalized Rank-Order Weighting). The weight $v_i(z, \underline{x})$ on the ith poorest person's income shortfall from the poverty line is given by:

$v_i(z, x) = (q+1-i)^{\lambda}$, where λ is any non-negative real number. Kakwani (1980) has proved a theorem to the effect that Axioms 1,2' and 3 together imply a class of poverty measures given by:

$$P_K(\lambda) = \frac{q}{nz\sum_{i\in S^P} i^{\lambda}} \sum_{i\in S^P} (z-x_i)(q+1-i)^{\lambda}.$$

It can be shown that for $\lambda=1$, $P_K(\lambda)=P_S$, and both Axioms TS-I and TS-II are violated. However, for $\lambda>1$, Axiom TS-I will be satisfied and for λ sufficiently large, Axiom TS-II will also be satiafied by the measure $P_K(\lambda)$. (For details, see Kakwani(1980)).

It might be worthwhile pointing out that there could be a 'pragmatic' (though by no means <u>formal</u>) difficulty with the class of poverty measures proposed by Kakwani. This has to do with the problematic nature of conferring an intuitively unambiguous interpretation on the parameter λ. Kakwani(1984,p.267) says: 'The value of λ may be chosen according to the society's preference for the sensitivity of the measure to an income transfer at different income positions.' One can foresee that society might experience some little difficulty in grasping the precise significance of λ well enough to be able to define a (non-arbitrary) preference ordering over, say, $\lambda=3.09$ and $\lambda=2.97$. (A similar problem arises with the choice of the 'index of inequality aversion' in Atkinson's (1970) ethical measure of inequality). A purely practical convenience with the poverty measure P is that it is quite responsive to the transfer-sensitivity axioms <u>without</u> confronting society

with the need to make choices the intuituve basis of which is not always crystal clear.

The attractive responsiveness of the poverty measure P to Kakwani's two transfer-sensitivity axioms arises from the implicit weighting structure on individual incomes in the inequality measure D^P. To this measure of inequality we now very briefly turn.

6. A MEASURE OF INEQUALITY.

It is routine to note that if we replace the poverty line z by the mean income μ of the entire population and let the head-count ratio H go to unity, them the poverty measure P collapses to a measure of inequality D which is given by

$$D = 1 - \sum_{i \in S} \mu_i / n\mu.$$

An analogous result for the Sen index of poverty and the Gini index of inequality is available in Sen(1976)).

It is useful to compare D with the Gini index of inequality G. As is well-known, G is obtained as a ratio of the weighted sum of the distance of each person's income from the mean, to the same sum for the extreme case of concentration in which a single person appoopriates the entire income; further, the weight on the i th person's income gap is the rank (n+1-i) of the ith person in the income vector:

$$G = \sum_{i \in S} (\mu - x_i)(n+1-i) / \mu \{ \sum_{i \in S} (n+1-i) - n \}$$

$$\cong 1 - 2 \sum_{i \in S} x_i (n+1-i) / n^2 \mu \quad \text{for large } n. \qquad \ldots(4)$$

By contrast, the inequality measure D is obtained as follows. First, for each person we obtain the mean income of all those with incomes less than or equal to the person's, and find the distance of this mean income from the mean income of the entire population; this distance, summed over all individuals, and expressed as a ratio of the same sum for the extreme case of concentration in which a single person appropriates the entire income yields D:

$$D = \sum_{i \in S} (\mu - \mu_i) / (n-1)\mu$$

$$\cong 1 - \sum_{i \in S} \mu_i / n\mu \quad \text{for large } n$$

$$= 1 - \sum_{i \in S} x_i \left(\sum_{j=i}^{n} j^{-1} \right) / n\mu. \qquad \ldots(5)$$

[3]

Letting v_i stand for the weight on the i th person's income in the expression for G, and w_i[4] for the weight on the ith person's income in the expression for D, one notes from (4) and (5) that

$$v_i - v_{i+1} = 2/n^2\mu; \text{ and} \qquad \ldots(6)$$

$$w_i - w_{i+1} = 1/in\mu.$$

As can be seen from (6), D has the desirable property that the difference in the weights of any pair of consecutive income levels declines as we move up the income vector --

unlike in the case of G where this difference is a constant. It is this latter property of G which accounts for the fact that so long as the area enclosed by the Lorenz curve and the diagonal of the unit square is the same for any set of intersecting Lorenz curves, G will rank the distributions represented by the different Lorenz curves indifferently -- without regard for whether the curves are skewed toward (0,0) or (1,1) of the unit square.

The inequality index D mitigates the severity of the problem with the Gini discussed above, by betraying a relatively greater sensitivity to inequality among the lower income groups. For this reason, the substitution of G^P by D^P in the variant P of the Sen index of poverty serves the purpose of making the poverty measure relatively more sensitive to the share in income of the poorer among the poor.

7. CONCLUDING REMARKS.

In this note, a variant P of Sen's index of poverty P_S has been derived, using the broad framework of Sen's approach, but with a different axiomatic system. This system has benefited from Nayak's (1985) reformulation of Sen's 'normalization' axiom. Unlike the Sen index, but like the class of poverty measures advanced by Kakwani, the index P fares rather well in respect of 'transfer-sensitivity' properties. This is attributable to the use of an

inequality measure D -- different from the Gini ratio G -- in the poverty index. The measure D incorporates a relatively more 'egalitarian' weighting structure on individual incomes than does the Gini. Finally, the poverty measure $\bar{P}$ has the practical advantage of being rather more easily interpretable than the class of poverty measures proposed by Kakwani.

NOTES.

1. Nayak (1985, p.6).

2. It should be noted that the poverty measure $P = H\{I+(1-I)D^P\}$ can be derived by using Axiom 3 just as well as by using Axiom α; it is just that the latter Axiom seems to capture the notion of normalization more appropriately.

3. It is readily verified that $D \in [0,1]$.

4. From (5) it can be seen that the weight on the ith person's income is simply the (divergent) harmonic series (1+1/2+1/3+...+1/n) less the first (i-1) terms of the series.

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A STANDARD-OF-LIVING RELATED INDEX OF AGGREGATE WELFARE*

S.SUBRAMANIAN

ABSTRACT: This note develops an index of 'nonpoorness' in an elementary, symmetric extension of the Sen index of poverty. The sum of the indices of poverty and nonpoorness provides a simple, aggregative measure of welfare related to the standard-of-living via the poverty line. This index is closely related to the Sen measure of 'real national income', and is used to evaluate the standards of rural and urban welfare in the State of Tamil Nadu.

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A. INTRODUCTION:

This note is concerned with proposing a simple, summary index of welfare which is related to standard-of-living in a community. The basic idea is to develop an index of 'nonpoorness' (N) as a symmetric countrpart to Sen's (1976a) index of poverty (P). While N can have some independent interest of its own as a description of the condition of the nonpoor (just as P is of considerable interest as a description of the condition of the poor), N and P together can serve the purpose of a normalized aggregative measure of welfare of the community as a whole. This measure -- which we may call W -- is closely related to Sen's (1976b) suggested measure of 'real national income', $\mu(1-G)$, where μ is *per capita* income and G is the Gini ratio of inequality in the distribution of income.

B. AN INDEX OF NONPOORNESS.

To construct an index of nonpoorness, N, all one has to do is hold a mirror to Sen's index of poverty, P. Under the circumstances, it should not be surprising that this section is entirely dependent on Sen (1976a).

$Y = (y_1, \ldots, y_i, \ldots, y_n)$ is an n-vector of incomes, with y_i standing for individual i's income. It will be assumed, without loss of generality, that $y_i \leq y_{i+1}$, $\forall i$. The poverty line (which is a mormatively determined income level below which people are judged to be absolutely impoverished) is denoted by z. The community of n individuals is divided into two groups -- the poor, with incomes less than or equal to z, who number q, and the nonpoor, with incomes exceeding z, who number (n-q). In what follows, q and (n-q) will be taken to be large enough to permit the approximation of (q+1) to q, and of (n-q+1) to (n-q).

$g_i \equiv (z-y_i)$ represents the gap between the poverty line z and the ith (nonpoor) person's income y_i ($i = q+1, \ldots, n$). Thus, $(-)g_i$ is the 'surplus' of the ith person's income over the poverty line. The aggregate income gap of the nonpoor, as a proprtion of the poverty line, is given by $I^N = \sum_{q+1}^{n} g_i/z = 1-\mu^N/z$, where μ^N is the *per capita* of the nonpoor. (Note that I^N is non-positive, so that a smaller I^N reflects a relatively larger mean income of the nonpoor. $H^N \equiv (n-q)/n$ is the proprtion

of the nonpoor in total population, and G^N is the Gini ratio of inequality in the distribution of income among the nonpoor.

A general expression for the index of nonpoorness is given by:

$$N = A(n,n-q,z)\sum_{q+1}^{n} v_i(z,Y)g_i, \qquad \ldots(1)$$

i.e., N is expressed as a weighted summation of the income gaps g_i of the nonpoor. $v_i(z,Y)$ is the weight attached to the income gap of the ith person, and $A(n,n-q,z)$ is a normalizing parameter. Two axioms which we can require N to satisfy are the following. (For a detailed discussion, see Sen(1976a,1981: Appendix C)).

Normalized Nonpoorness (Axiom N). In the special case in which y_i = (say) y^* ∀i (so that g_i = (say) g^* ∀i) the index of nonpoorness is given by:

$N = H^N I^N$.

Rank-Order Weighting (Axiom R). On the right hand side of (1), the weight $v_i(z,Y)$ on the ith person's income gap g_i is given by:

$v_i(z,Y) = n+1-i$.

The following theorem can now be stated.

THEOREM. For large (n-q), the only measure of nonpoorness which satisfies Axioms N and R is given by

$N = H^N\{I^N+(1-I^N)G^N\}$.

PROOF. Invoking Axiom R in the context of the special case in which $g_i = g^* \forall i$, (1) can be written as

$$N = A(n,n-q,z)\,g^* \sum_{q+1}^{n} (n+1-i)$$

$$\cong A(n,n-q,z)\,g^*(n-q)^2/2. \qquad \ldots(2)$$

By Axiom N, when $g_i = g^* \forall i$,

$$N = \left(\frac{n-q}{n}\right)\left(\frac{g^*}{z}\right). \qquad \ldots(3)$$

From (2) and (3) one obtains

$$A(n,n-q,z) = 2/(n-q)nz. \qquad \ldots(4)$$

From (1), (2) and (4), we have

$$N = \frac{2}{(n-q)nz}\left\{\sum_{q+1}^{n}(z-y_i)(n+1-i)\right\}$$

which, upon simplification, can be written as

$$N = \frac{n-q}{n} - 2\sum_{q+1}^{n}(n+1-i)y_i/(n-q)nz. \qquad \ldots(5)$$

Note, next, that the Gini ratio of inequality in the distribution of income among the nonpoor can, for large $(n-q)$, be written as

$$G^N \cong 1 - 2\sum_{q+1}^{n}(n+1-i)y_i/(n-q)^2\mu^N, \text{ so that}$$

$$2\sum_{q+1}^{n}(n+1-i)y_i = (1-G^N)(n-q)^2\mu^N. \qquad \ldots(6)$$

Substituting for $2\sum_{q+1}^{n}(n+1-i)y_i$ from (6) into (5), and simplifying, yields

$$N = \left(\frac{n-q}{n}\right) - \frac{(1-G^N)(n-q)\mu^N}{nz}. \qquad \ldots(7)$$

Recalling that $(n-q)/n \equiv H^N$ and $1-\mu^N/z \equiv I^N$, (7) can be

rewritten as

$N = H^N\{I^N+(1-I^N)G^N\}$. □

C. SOME PROPERTIES OF N .

(i). It is clear that $N \leq 1$. The term in curly brackets $\{I^N+(1-I^N)G^N\}$ in the expression for N is maximized at $G^N=1$, and minimized at $G^N=0$. For $G^N=1$, $N=H^N \varepsilon(-\infty, 1]$.

(ii). N is non-decreasing in I^N and G^N, and non-increasing im H^N; $\frac{\delta N}{\delta I^N} = H^N(1-G^N) \geq 0$; $\frac{\delta N}{\delta G^N} = H^N(1-I^N) \geq 0$; and $\frac{\delta N}{\delta H^N} = I^N+(1-I^N)G^N \leq 0$. (To see that $\frac{\delta N}{\delta H^N} \leq 0$, or equivalently, that $G^N \leq -I^N/(1+I^N)$, note that $G^N \stackrel{\circ}{=} 1-2\sum_{q+1}^{n}(n+1-i)y_i/(n-q)^2\mu^N$. Since $y_i \geq z \forall i$, $G^N \leq 1 - 2z\sum_{q+1}^{n}(n+1-i)/(n-q)^2\mu^N$, that is,

$G^N \leq 1 - \frac{2z(n-q)^2/2}{2\mu^N(n-q)^2} = 1-z/\mu^N \stackrel{\circ}{=} -I^N/(1+I^N)$).

(iii). The smaller the value of N, the greater the degree of nonpoorness.

D. AN AGGREGATE STANDARD-OF-LIVING RELATED WELFARE INDEX

In obvious notation, the Sen index of poverty can be written as:

$P = H^P\{I^P+(1-I^P)G^P\}$, with $P \varepsilon [0,1]$.

A simple measure of aggregate welfare, W, is obtained by adding the index of poverty and the index of nonpoorness:

$$W = P+N \ . \qquad \ldots(8)$$

Some manipulation will show (See Anand(1977)) that P and N can be written, respectively, as

$$P = H^P\{1-\frac{\mu^P}{z}(1-G^P)\} \text{ and } N = H^N\{1-\frac{\mu^N}{z}(1-G^N)\} \qquad \ldots(9)$$

Notice that $1-\frac{\mu^P}{z}(1-G^P)$ is the prportionate shortfall of the mean income of the poor, adjusted for distributuonal considerations, from the poverty line, while $1-\frac{\mu^N}{z}(1-G^N)$

is the (negative of the) proportionate surplus of the mean income of the nonpoor, adjusted for distributional considerations, from the poverty line. Designating the two proportionate income gaps by $\tilde{I}^P$ and $\tilde{I}^N$ respectively, we obtain:

$$W = H^P\tilde{I}^P + H^N\tilde{I}^N \ .$$

The aggregate standard-of-living related index of welfare W can thus be seen simply as the sum of the 'distributionally adjusted' proportionate income gaps of the poor and the nonpoor, weighted, respectively, by the proportionate size of each of the two groups. It is readily clear that $W \in (-\infty, 1]$, with a smaller value of W reflecting a relatively greater level of aggregate welfare.

To see that the measure W is a close relative of Sen's (1976b) measure of real national income $\mu(1-G)$, note from (8) and (9) that as either H^P or H^N goes to unity, W collapses to

$$W = 1 - \frac{\mu(1-G)}{z} \qquad \ldots(10)$$

(10) reveals that W is simply a linear transform of the Sen measure of real national income $\mu(1-G)$: in the special case in which every individual is considered to be poor (or nonpoor), the index W reflects the proportionate distance of the real income measure $\mu(1-G)$ from the poverty line z.

E. <u>AN APPLICATION TO TAMIL NADU STATE</u>.

Using various Rounds of the National Sample Survey data on distribution of consumer expenditure, we have calculated the Sen index of poverty (P), the index of nonpoorness (N), and the standard-of-living related index of aggregate welfare (W) for selected years in the 'sixties and the 'seventies in the State of Tamil Nadu. The exercise has been done for ten years -- 1961-62, 1963-64, 1965-66, 1967-68, 1968-69, 1969-70, 1970-71, 1972-73, 1973-74, and 1977-78 -- separately for rural and urban Tamil Nadu. For the rural areas, we have used as the poverty line a consumer expenditure level of Rs.15 *per capita* for a period of thirty days at 1960-61 prices, and for the urban areas a consumer expenditure level of Rs.20 *per capita* for a period of thirty days at 1960-61 prices. To obtain the poverty lines at current prices, we have used the Consumer Price Index of Agricultural Labourers (CPIAL) as the appropriate price deflator for rural Tamil Nadu, and the Consumer Price Index of Industrial Workers (CPIIW) as the

price deflator for urban Tamil Nadu. (The CPIIW has beem obtained as a simple average of the Indices reported for four Centres in urban Tamil Nadu by the Indian Labour Journal). We shall not here enter into a critique of the NSS data on consumer expenditure distribution: the interested reader should consult Vaidyanathan (1983).

The results of our exercise are presented in Tables 1 and 2. If we accept the NSS data on consumer expenditure at face value, then the following points emerge from a consideration of the information presented in the two tables:

(i) In the rural areas of Tamil Nadu, the Sen index of poverty falls below its 1961-62 level in only one year thereafter in the 'sixties -- 1963-64 -- and in four years thereafter in the 'seventies -- 1970-71, 1972-73, 1973-74 and 1977-78. In the urban areas, there is not a single year after 1961-62 when the Sen index of poverty falls below its 1961-62 level.

(ii) In rural Tamil Nadu, the index of nonpoorness (N) falls below its 1961-62 level in only two years thereafter -- 1973-74 and 1977-78 -- while in the urban areas, again one notes that the index at no time after 1961-62 falls below its 1961-62 level.

(iii) Insofar as the normalized index of aggregative welfare (W) is concerned, in rural Tamil Nadu this index falls below its 1961-62 level in only four years thereafter

Table-1: Indices of Poverty, nonpoorness and aggregate welfare for Rural Tamil Nadu (1961/62-1977/78)

Year	Head-Count Ratio of Poor (H^P)	Proportionate Expenditure Gap of Poor (I^P)	Gini index of inequality in Distribution of Expenditure among poor (G^P)	Sen index of Poverty (p)	Head-Count Ratio of Non-poor (H^N)	Proportionate Expenditure Gap of Non-poor (I^N)	Gini Index of Inequality in Distribution of Expenditure among Nonpoor (G^N)	Index of non-poorness (N)	Index of Aggregate Welfare (W) [W=P+N]
1961/62	.443	.299	.154	.180	.557	-.745	.228	-.193	-.013
1963/64	.452	.276	.133	.162	.548	-.744	.232	-.187	-.025
1965/66	.534	.294	.135	.208	.466	-.648	.206	-.144	.064
1967/68	.513	.280	.130	.191	.487	-.597	.220	-.120	.071
1968/69	.541	.328	.154	.234	.459	-.671	.205	-.151	.083
1969/70	.556	.309	.144	.227	.444	-.671	.222	-.133	.094
1970/71	.497	.268	.127	.179	.503	-.560	.184	-.137	.042
1972/73	.432	.251	.120	.147	.568	-.662	.199	-.187	-.040
1973/74	.416	.265	.117	.146	.584	-.715	.189	-.229	-.083
1977/78	.402	.264	.136	.146	.598	-.813	.259	-.205	-.059

Table-2: Indices of Poverty, Nonpoorness and Aggregate Welfare for Urban Tamil Nadu (1961/62-1977/78)

Year	Head-Count Ratio of Poor (H^P)	Proportionate Expenditure Gap of Poor (I^P)	Gini Index of inequality in Distribution of Expenditure among poor (G^P)	Sen Index of Poverty (P)	Head-Count Ratio of Non-poor (H^N)	Proportionate Expenditure Gap of Non-poor (I^N)	Gini Index of Inequality in Distribution of Expenditure among Nonpoor (G^N)	Index of nonpoorness (N)	Index of Aggregate Welfare (W) [$W=P+N$]
1961/62	.434	.270	.131	.159	.566	-.903	.285	-.204	-.045
1963/64	.482	.275	.128	.177	.518	-.879	.288	-.175	-.002
1965/66	.528	.323	.147	.223	.472	-.815	.247	-.173	.050
1967/68	.504	.295	.142	.199	.496	-.959	.324	-.161	.038
1968/69	.493	.305	.144	.200	.507	-.824	.250	-.187	.013
1969/70	.494	.298	.142	.197	.506	-.836	.268	-.175	.022
1970/71	.534	.307	.140	.216	.466	-.844	.273	-.159	.057
1972/73	.551	.308	.139	.223	.449	-.726	.234	-.145	.078
1973/74	.586	.312	.137	.238	.414	-.687	.227	-.126	.112
1977/78	.462	.289	.139	.179	.538	-.862	.265	-.199	-.020

(1963-64, 1972-73, 1973-74 and 1977-78) while for urban Tamil Nadu, 1961-62 emerges as the year with the highest standard of aggregate welfare. In five years in the rural areas -- 1965-66, 1967-68, 1968-69,1969-70 and L970-71 -- and seven years in the urban areas -- 1965-66, 1967-68, 1968-69, 1969-70, 1970-71, 1972-73 and 1973-74 -- the index of aggregate welfare has been strictly positive; from this we must infer that in these years, the welfare level has been lower than what it would have been if <u>every person's expenditure level had just been equal to the poverty line level</u>; this speaks for very low levels of welfare indeed.

(iv) From the preceding three points, one notes that in terms of poverty, or nonpoorness, or aggregate welfare, the 'seventies seem to represent an improvement over the base year (1961-62) of our reference period in the rural areas, whereas for the urban areas no such heartening feature is discernible.

(v) As <u>between</u> the rural and the urban areas, the urban Sen index of poverty exceeds the rural Sen index in seven of the ten years for which we have information; these years are 1963-64, 1965-66, 1967-68, 1970-71, 1972-73, 1973-74 and 1977-78. For the six years in the 'sixties for which we have data, the average of the Sen index series works out to 0.193 and 0.200 in the urban and rural areas respectively, and the ratio of the two averages is 0.97; for the four years in the 'seventies, theaverages of the

Sen index series work out to 0.214 and 0.155 in the urban and the rural areas respectively, and the ratio of the two averages is 1.38. The disparity in poverty as between the rural and the urban areas seems to be loaded against the urban areas during the 'seventies.

(vi) In six years out of ten the index of nonpoorness is lower in the urban areas than in the rural areas of Tamil Nadu.(To repeat: A lower value of N reflects a greater degree of prosperity). These six years are: 1961-62, 1965-66, 1967-68, 1968-69, 1969-70 and 1970-71. In three of these years -- 1965-66, 1967-68 and 1970-71 -- the urban Sen index (of poverty) has been higher than the rural Sen index. Thus, both greater prosperity and greater poverty seem to characterize the urban areas vis a vis the rural.

(vii) The index of aggregate welfare is higher (meaning that the standard of welfare is lower) in the urban areas than in the rural areas in five years (1963-64, 1970-71, 1972-73, 1973-74 and 1977-78). It is in the 'seventies that one notices a particularly pronounced lower level of aggregate welfare in the urban areas relative to the rural ares.

(viii) In sum, we discern no heartening trend over time of an orderly dimunition in poverty, or increase in nonpoorness or aggregate welfare in either the rural or the urban areas of Tamil Nadu. The data also suggest that poverty and low levels of welfare are not -- at least in Tamil Nadu -- predominantly rural phenomena.

F. CONCLUDING OBSERVATIONS.

(i) In this note, we have developed an index of nonpoorness (N), in a routine extension of the Sen index of poverty (P). N lies in the interval $(-\infty, 1]$. N could have some independent interest as a compact description of the condition of the nonpoor, just as P is of considerable interest as a summary description of the condition of the poor.

(ii) N can also be combined with P to yield a more comprehensive picture of aggregate welfare. How one chooses to combine information on N and P would be a function of one's descriptive objective. An example of one kind of descriptive use to which information on N and P can be put is the following. For two regions 1 and 2, if $N_1 < N_2$ and $P_1 > P_2$, then we can infer that not only is the condition of the poor absolutely worse in region 1 than in region 2, but that the condition of the poor, in relation to that of the nonpoor, is worse in region 1 than in region 2.

(iii) A simple aggregative measure of welfare (W) which is related to the standard-of-living via the poverty line, is obtained by adding the indices of poverty and nonpoorness: $W = P+N$. In the special case where all individuals are regarded as poor (or nonpoor), the index W can be seen to be very closely related to the Sen measure of 'real national income'; in the more general case, W shows the proportionate distance of each of the poor and the nonpoor group's real income from the poverty line, weighted, respectively, by the incidence

(contd. overleaf...)

of poverty and nonpoorness. A possible virtue of this index is that it is a pure number: in performing spatial or inter-temporal comparisons, problems of correcting for price-variations across space or over time are minimized.

(iv) In an application to the State of Tamil Nadu, we have computed P, N and W in selected years in the 'sixties and the 'seventies for both rural and urban Tamil Nadu. The results have been discussed in section E.

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ON CONSTRUCTING AN INDEX OF DISCRIMINATION : A BRIEF NOTE

S.Subramanian*

ABSTRACT: In this note, we derive a simple summary statistic which measures the extent of discrimination, or asymmetry in access, of groups classified by one attribute to groups classified by another. Ramachandran (2) discusses the phenomenon of asymmetries of access to different economic classes experienced by persons of different castes in the context of a Tamil Nadu village. The present paper extends the (purely technical aspect of the) analysis in (2); a more rudimentary version of the index of discrimination discussed in this note is also to be found in Radhakrishnan (1).

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ON CONSTRUCTING AN INDEX OF DISCRIMINATION: A BRIEF NOTE

S.Subramanian

The phenomenon of discriminatory barriers to entry is one which is experienced -- in a variety of social and economic situations -- by those groups of individuals who are relatively'disadvantaged' or 'underprivileged' in the socio-economic hierarchy of the reference society. Common examples would include discrimination in the job market as between males and females or blacks and whites, discrimination in access to education as between the affluent and the poor, discrimination in access to occupational categories (or, more broadly, 'economic classes') as betwuen 'forward caste' and 'backward caste' persons, and so on.

The examples can be readily multiplied, but what they have in common is that if a given set of individuals is classified according to two broad attributes -- call these A and B -- then one perceives an asymmetry in the relative access of groups classified by A to groups classified by B. A quantitative measure of such an asymmetry of access would serve the purpose of a very useful summary statistic. The object of this note is to develop -- briefly and in a preliminary sort of way -- one such index of 'inter-group asymmetry of access' or, more compactly, 'discrimination', which we shall call D.

In what follows we shall, for specificity, employ

'class' (or 'occupational category') and 'caste' as the two attributes according to which individuals are classified.

Consider an m x n matrix (C_{ij}) with C_{ij} standing for the number of persons of caste i (i=1,...,m) belonging to occupational category j (j=1,...,n). For all i, define $C_i \equiv \sum_j C_{ij}$, and for all j, define $C_j \equiv \sum_i C_{ij}$. C_i and C_j are, of course, the populations of caste i and of occupational category j, respectively. The total population of the society is $P \equiv \sum_i C_i \equiv \sum_j C_j$.

It seems reasonable to suggest that if membership in any given caste has no influence on the prospects of membership to any given occupational category, then persons belonging to any given caste should be distributed among the various occupational categories in the same proportion which the population of the relevant caste bears to total population. That is, one could envisage an 'ideal' or 'normative' matrix representing a case of 'perfect symmetry in access' given by (C^*_{ij}), with $C^*_{ij} \equiv \frac{C_i}{P} \cdot C_j$, i=1,...,m; j=1,...,n.

Next, consider the matrix (δ_{ij}) , with $\delta_{ij} \equiv \left| C_{ij}/C^*_{ij} - 1 \right|$, i=1,...,m and j=1,...,n. δ_{ij} is the proportionate deviation (in absolute terms) of C_{ij}

from its 'normative' value C^*_{ij}. If we were to sign δ_{ij}, then $\delta_{ij}>0$ would imply that caste i is 'over-represented' in occupational category j, and $\delta_{ij}<0$ would imply that caste i is 'under-represented' in occupational category j. The matrix (δ_{ij}) thus presents a comprehensive picture of the extent of access which each caste has to each occupational category.

To obtain a more aggregative picture, define $D_i \equiv \sum_j \delta_{ij}$, i=1,...,m. D_i can be viewed as an index of asymmetry of access of caste i, or, simply, as caste i's index of discrimination. Notice that if $C_{ij} = C^*_{ij}\ \forall j$, then $\delta_{ij}=0\ \forall j$ and $D_i=0$. Thus, the case of perfect symmetry of access for caste i would correspond to $D_i=0$ -- which is, of course, the minimum value that D_i can assume. (It should be clear now why we have chosen to present the δ_{ij} in absolute terms; if we were to sign the δ_{ij}, then a high positive value for some of the δ_{ij} and a high negative value for the remaining δ_{ij} could, upon algebraic summation, yield a value of D_i which is close to zero; to interpret from this that caste i has symmetric access could be very misleading).

We now have two m-vectors: the vector $\underline{D}$ = $(D_1,\ldots,D_m)$, which is the vector of actual indices of discrimination for each caste i=1,...,m;[1] and the zero-

vector $\underline{0} = (0,\ldots,0)$ which is the vector that would correspond to perfect symmetry of access ('no discrimination') for each caste $i = 1,\ldots,m$. For any two vectors $\underline{x}$ and $\underline{y}$ in R^m_+ (the nonnegative orthant of m-dimensional real space), if $\|\underline{x} - \underline{y}\| = \{\sum_{i=1}^{m}(x_i-y_i)^2\}^{1/2}$ is the distance between the two vectors, then we can write a simple, normalized, aggregative index of discrimination for the community as a whole as

$$D = \|\underline{D} - \underline{0}\| / \|\tilde{\underline{D}} - \underline{0}\| \text{ where } \tilde{\underline{D}} \text{ is that } \underline{D} \text{ which maximizes } \|\underline{D}-\underline{0}\| \quad \ldots(1)$$

In (1) we are expressing the aggregative index of discrimination D as the ratio of the distance of $\underline{D}$ from $\underline{0}$ to the maximum possible distance of $\underline{D}$ from $\underline{0}$. It is easy to check that the index has the convenient property that $D \in [0,1]$, with a higher value of D signifying a relatively greater extent of asymmetry of access.

It remains to specify how $\tilde{\underline{D}}$ is to be derived. Given any (C_{ij}) matrix, we have seen how the vector $\underline{D}$ can be derived. The vector $\tilde{\underline{D}}$ can be seen to be the $\underline{D}$ vector that corresponds to a particular type of (C_{ij}) matrix. Before coming to that, we take note of one heroic assumption which we need to make -- namely that m ('the number of castes') is equal to n ('the number of occupational categories'). While this amounts to a somewhat restrictive requirement, it should generally be possible -- through

suitable grouping of castes and occupational categories -- to derive a classificatory scheme on which one can 'force' an equality between the number of castes and the number of occupational categories. Given, then, that m=n, $\underset{\sim}{\tilde{D}}$ can be seen as the $\underset{\sim}{D}$-vector corresponding to a (C_{ij}) matrix for which the following is true:

$$\left.\begin{aligned} C_{ii} &= C_i, \quad i=1,\ldots,m; \text{ and} \\ C_{ij} &= 0, \quad i \neq j \end{aligned}\right\} \qquad \ldots(2)$$

With the above allocation mechanism, it is readily verified that the δ_{ij} -- which we shall call $\tilde{\delta}_{ij}$ for a (C_{ij}) matrix of the type described in (2) -- are given by:

$$\tilde{\delta}_{ij} = \begin{cases} P/C_i - 1 & \text{for } i=j \\ 1 & \text{for } i \neq j \end{cases} \qquad \ldots(3)$$

It is routine to note next that

$$\tilde{D}_i \equiv \sum_{j=1}^{m} \tilde{\delta}_{ij} = (m - 2 + P/C_i), \quad i=1,\ldots,m; \text{ and}$$

$$\|\underset{\sim}{\tilde{D}}-\underset{\sim}{0}\| = \left(\sum_{i=1}^{m}\tilde{D}_i^2\right)^{1/2} = \left\{\sum_{i=1}^{m}(m-2+P/C_i)^2\right\}^{1/2} \qquad \ldots(4)$$

A very informal rationale for the procedure described above for deriving $\underset{\sim}{\tilde{D}}$ is available from an examination of a simple specialized case -- that of a square matrix of order 2. The procedure is simply a generalization of the result for a 2x2 matrix: the following is, therefore, at best a suggestive demonstration and not a formal proof of the claim that $\underset{\sim}{\tilde{D}}$ can be derived from a (C_{ij}) matrix

for which $C_{ij} = C_i$ for $i=j$ and $C_{ij} = 0$ for $i \neq j$.

Consider, then a situation in which m=n=2, viz., there are two castes of sizes C_1 and C_2 respectively, and two occupational categories of sizes O_1 and O_2 respectively, Let α and β denote, respectively, the proportion of people of caste 1 in occupational category 1 and the proportion of people of people of caste 2 in occupational category 1. The quantities $(1-\alpha)$, $(1-\beta)$ can be similarly interpreted. That is, $\alpha = C_{11}/C_1$; $(1-\alpha) = C_{12}/C_1$; $\beta = C_{21}/C_2$ and $(1-\beta) = C_{22}/C_2$. The values of D_1 and D_2 will then be given by:

$$D_1 = \left|\frac{C_{11} \cdot P}{C_1(C_{11}+C_{21})} - 1\right| + \left|\frac{C_{12} \cdot P}{C_1(C_{12}+C_{22})} - 1\right|$$

$$= \left|\frac{\alpha P}{\alpha C_1+\beta C_2} - 1\right| + \left|\frac{(1-\alpha)P}{(1-\alpha)C_1+(1-\beta)C_2} - 1\right|; \text{ and}$$

$$D_2 = \left|\frac{C_{21} \cdot P}{C_2(C_{11}+C_{21})} - 1\right| + \left|\frac{C_{22} \cdot P}{C_2(C_{11}+C_{21})} - 1\right|$$

$$= \left|\frac{\beta P}{\alpha C_1+\beta C_2} - 1\right| + \left|\frac{(1-\beta)P}{(1-\alpha)C_1+(1-\beta)C_2} - 1\right|.$$

We are interested in maximizing $\|\underline{D}-\underline{0}\| = \{\sum_{i=1}^{2}(D_i^2)\}^{1/2}$ by suitable choice of α and β; the problem can be written as one in constrained optimization of the following type:

$$\underset{\alpha,\beta}{\text{Maximize}}\ V = \{\{\left|\frac{\alpha P}{\alpha C_1+\beta C_2} - 1\right| + \left|\frac{(1-\alpha)P}{(1-\alpha)C_1+(1-\beta)C_2} - 1\right|\}^2$$

$$+\{\left|\frac{\beta P}{\alpha C_1+\beta C_2} - 1\right| + \left|\frac{(1-\beta)P}{(1-\alpha)C_1+(1-\beta)C_2} - 1\right|\}^2\}^{1/2}$$

subject to:

$$1-\alpha \geqq 0;$$

$$1-\beta \geqq 0; \qquad \ldots(5)$$

$$\alpha,\beta \geqq 0.$$

Let $C_1 \equiv qP$ and $C_2 \equiv (1-q)P$, with $q\epsilon(0,1)$. Then the maximand in (5) can be written as:

$$V = \left\{\left\{\left|\frac{\alpha}{q\alpha+(1-q)\beta} - 1\right| + \left|\frac{1-\alpha}{q(1-\alpha)+(1-q)(1-\beta)} - 1\right|\right\}^2 + \left\{\left|\frac{\beta}{q\alpha+(1-q)\beta} - 1\right| + \left|\frac{1-\beta}{q(1-\alpha)+(1-q)(1-\beta)} - 1\right|\right\}^2\right\}^{1/2}$$

which, upon simplification, reduces to

$$V = \left\{\frac{(\alpha-\beta)^2\{(1-q)^2+q^2\}}{\{q\alpha+(1-q)\beta\}^2\{q(1-\alpha)+(1-q)(1-\beta)\}^2}\right\}^{1/2}. \qquad \ldots(6)$$

Our claim now is that $(\alpha,\beta) = (1,0)$ maximizes V. Putting $\alpha = 1$ and $\beta = 0$ in the right hand side of (6), we obtain

$$V = \{(1-q)^2+q^2\}^{1/2}/q(1-q) \ . \qquad \ldots(7)$$

Suppose now that $(\alpha,\beta) = (1,0)$ is not an optimal solution. Then there must exist an $(\alpha^*,\beta^*) \neq (1,0)$ such that, in view of (6) and (7),

$$\frac{(\alpha^*-\beta^*)}{\{q\alpha^*+(1-q)\beta^*\}\{q(1-\alpha^*)+(1-q)(1-\beta^*)\}} > 1/q(1-q),$$

which condition can, through suitable manipulation and rearrangement, be written as the requirement

$$-2\{2q(1-q)\beta^*(1-\alpha^*)+q^2\alpha^*(1-\alpha^*)+(1-q)^2\beta^*(1-\beta^*)\} > 0 \quad \ldots(8)$$

Since $0<q<1$, $0\leq\alpha^*\leq 1$ and $0\leq\beta^*\leq 1$, it is immediately apparent that (8) cannot be true. Hence $(\alpha,\beta) = (1,0)$ is an optimal solution, as desired. Generalizing from this specific example, one obtains the algorithm provided in (2) to derive $\tilde{D}$.

For an illustration of a simple application of the index of discrimination D, consider the following. From data contained in the World Bank publication World Development Report (WDR) for 1984, one can obtain a comparitive picture, for different countries, of the extent of discrimination in access to enrolment in primary school as between males and females. For illustrative purposes, we confine our attention to two countries -- India and Czechoslovakia. The following are data for the year 1981 (it should be noted that the data are subject to all the caveats and limitations discussed in the Technical Note to the WDR (p.283)).

Country	Number enrolled in Primary School as Percent of Age Group.		
	Male	Female	Total
India	93	64	79
Czechoslovakia	90	91	90

Source: From WDR 1984(Table 25,pp.266-267)

Letting F stand for the total population of females in the relevant age group and M for the total population of males in the relevant age group, we have for India:

$$\frac{.93M + .64F}{M + F} = .79, \text{ from which one obtains}$$

$M = 1.07F.$

The (C_{ij}) matrix -- with C_{ij} being interpreted as the number of members of sex i with primary school enrolment status j -- can now be written as follows, after expressing all populations in terms of female population:

		Primary School Enrolment Status		
		Enrolled in Primary School	Not Enrolled in Primary School	Total
Sex	Male	.955F	.075F	1.07F
	Female	.640F	.360F	F
	Total	1.635F	.435F	2.07F

Next, recalling that $C^*_{ij} \equiv (C_i/F).C_j$, the elements of the (C^*_{ij}) matrix can be written as : C_{11}=.845F, C_{12}=.225F, C_{21}=.790F and C_{22}=.210F. The $\delta_{ij} \equiv |C_{ij}/C^*_{ij}-1|$ can then be computed, and it can be verified that δ_{11}=.1775, δ_{12} =.6670, δ_{21}=.190, and δ_{22}=.714. Noting that $\underline{D} \equiv (D_1, D_2)$ $= (\sum_{j=1}^{2}\delta_{1j}, \sum_{j=1}^{2}\delta_{2j})$, one obtains $\underline{D}$ = (.8445,.9040), and

$\|\underline{D} - \underline{0}\| = (\sum_{i=1}^{2} D_i^2)^{1/2} = 1.2371$. Finally, it is a routine matter to note from (4), and after making the appropriate substitutions from the data, that $\|\underline{\tilde{D}} - \underline{0}\| = \{\sum_{i=1}^{m}(m-2+P/C_i)^2\}^{1/2}$ $= 2.833$. One then obtains $D = \|\underline{D} - \underline{0}\| / \|\underline{\tilde{D}} - \underline{0}\| = 1.2371/2.833 \cong 0.44$. For Czechoslovakia, it is evident from inspection that $D \cong 0$. Thus, while there is hardly any discrimination as between males and females in enrolment to primary school in Czechoslovakia, in India the index of ddiscrimination is quite high at 0,44.

One point relating to the index of discrimination D deserves mention. A measure such as D enables us to compare all possible vectors $\underline{D}$. A different form of comparison would be through 'vector dominance'. Define the binary relation '$\gtrsim$' as: $\forall\ \underline{D}^1, \underline{D}^2$ (where $\underline{D}^1$ and $\underline{D}^2$ are equi-dimensional vectors) $\underline{D}^1 \gtrsim \underline{D}^2$ if and only if $\forall i$: $D_i^1 \geq D_i^2$. If $\underline{D}^1 \gtrsim \underline{D}^2$, we shall say that $\underline{D}^1$ 'weakly dominates' $\underline{D}^2$; and whenever $\underline{D}^1$ weakly dominates $\underline{D}^2$, it can be inferred that asymmetry of access, or discrimination, in the one case is at least as great as in the other. Now it is clear that the binary relation $\gtrsim$ is reflexive and transitive, but not necessarily complete: typically, $\gtrsim$ will be a <u>quasi-ordering</u>. The index D, however, overcomes this problem; as we have seen, one can define a complete ordering of all vectors $\underline{D}$ using the index D. It is,

however, important to emphasize that the 'completemess' feature of D is purchased at a price; that of the information lost in the process of aggregating the D_i and collapsing the index of discrimination into a single real number. The analogy, in this context, with the Lorenz distribution and the Gini ratio of inequality is immediate.

A final word of caution may be in order. The domain of applicability of a measure such as D for comparing asymmetries of access in, say, two different societies would depend crucially on whether there is, *prima facie*, a sufficiency of socio-economic similarities between the two societies to render such a comparison meaningful. Failure to engage in informed or socio-economically meaningful comparisons could result in an indiscriminate and misleading application of a measure such as D.

NOTE.

1. A slight variant of the vector D is, in effect, the index of asymmetry of access employed in (2).

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